

Harmonic Oscillator States with Non-Integer Orbital Angular Momentum

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Abstract

We study the quantum mechanical harmonic oscillator in two and three dimensions, with particular attention to the solutions as basis states for representing their respective symmetry groups — $O(2)$, $O(3)$, and $O(2,1)$. Solving the Schrodinger equation by separating variables in polar coordinates, we obtain wavefunctions characterized by a principal quantum number, the group Casimir eigenvalue, and one observable component of orbital angular momentum, with eigenvalue $m + s$, for integer m and real constant parameter s . For each of the three symmetry groups, s splits the solutions into two inequivalent representations, one associated with $s = 0$, from which we recover the familiar description of the oscillator as a product of one-dimensional solutions, and the other with $s > 0$ (in three dimensions, solutions are found for $s = 0$ and $s = 1/2$) whose solutions are non-separable in Cartesian coordinates, and are hence overlooked by the standard Fock space approach. In two dimensions, a single set of creation and annihilation operators forms a ladder representation for the allowed oscillator states for any s , and the degeneracy of energy states is always finite. However, in three dimensions, the integer and half-integer eigenstates are qualitatively different: the former can be expressed as finite dimensional irreducible tensors under $O(3)$ or $O(2,1)$ while the latter exhibit infinite degeneracy. Creation operators that produce the allowed integer states by acting on the non-degenerate ground state are constructed as irreducible tensor products of the fundamental vector representation. However, since the half-integer ground state has infinite degeneracy, the vector representation of the creation operators does not take this ground state to the calculated first excited level, and the general construction does not act as a ladder representation for the half-integer states. For all $s \neq 0$ solutions, the $SU(N)$ symmetry of the harmonic oscillator Hamiltonian recently discussed by Bars is spontaneously broken by the ground state. The connection of this symmetry breaking to the non-separability into one-dimensional Cartesian solutions is demonstrated.

1 Introduction

Along with its classical counterpart, the quantum harmonic oscillator is a well-studied model with exact solutions and connections to many physical systems for which it serves as foundation or approximation. Beyond its application to atomic and molecular spectra, statistical

mechanics, and by way of various relativistic generalizations to quark dynamics, certain general techniques associated with the harmonic oscillator, including the Fock space ensemble of uncoupled modes and Dirac's factorization of the Hamiltonian into creation and annihilation operators, serve as conceptual building blocks in areas ranging from blackbody radiation to canonical quantization and string theory. Yet, despite the subject's long history, fundamental new insights continue to emerge [1, 2].

In this paper, we study the quantum mechanical harmonic oscillator in two and three dimensions, with emphasis on the solutions as basis states for representations of their respective symmetry groups — $O(2)$, $O(3)$, and $O(2,1)$. The original motivation for this work was an attempt to develop a ladder representation of creation and annihilation operators for the relativistic oscillator model found by Horwitz and Arshansky [4] who applied a covariant formulation of quantum mechanics [5] to relativistic generalizations of the classical central force bound state problems. These models, which are obtained by inducing a representation of $O(3,1)$ on wavefunctions whose dynamics are restricted to the spacelike sector of an $O(2,1)$ -invariant subspace, exhibit a positive spectrum, and belong to half-integral representations of $O(3,1)$. According to a virial theorem [6] for the covariant quantum mechanics, the restriction to spacelike dynamics guarantees a positive spectrum, but since there is no obvious way to realize this nonholonomic constraint in Cartesian coordinates, the eigenvalue equation was posed in a hyperspherical parameterization. To address the unusual characteristics of these solutions, we sought to develop a creation/annihilation algebra associated with polar coordinates and non-integer orbital angular momentum. Although the algebraic approach succeeds in reproducing the basic oscillator features for both integer and non-integer representations in two dimensions and for the integer representations in three dimensions, Dirac's factorization of the Hamiltonian does not lead to creation/annihilation operators for the half-integer representations of $O(3)$ or $O(2,1)$. This paper presents a summary of results to be demonstrated in greater detail in a subsequent paper. To expose the common features of the oscillators associated with the three symmetries, we develop the models in tandem and use common notation, as far as is possible.

We write the harmonic oscillator Hamiltonian

$$H = \frac{1}{2} (p^2 + \omega^2 x^2) = \frac{1}{2} \eta_{\mu\nu} (p^\mu p^\nu + \omega^2 x^\mu x^\nu) \quad (1)$$

to describe either an $O(D)$ nonrelativistic oscillator with Euclidean metric

$$\eta_{\mu\nu} = \delta_{\mu\nu}, \quad \mu, \nu = 1, \dots, D \quad (2)$$

or an $O(D-1, 1)$ relativistic oscillator with Lorentz metric

$$\eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1), \quad \mu, \nu = 0, \dots, D-1. \quad (3)$$

The standard approach to Fock space proceeds by separation of Cartesian variables and subsequent application of Dirac's factorization of the one-dimensional Hamiltonian for each degree of freedom. Assuming a product solution of one-dimensional oscillators

$$\psi(x) = \prod_{\mu} \psi(x^{\mu}) \quad E = \sum_{\mu} E^{\mu} \quad (4)$$

the Hamiltonian separates into a sum of D mode-number terms as

$$H = \omega \eta_{\mu\nu} \left(\bar{a}^{\mu} a^{\nu} + \frac{1}{2} \eta^{\mu\nu} \right) = \omega \sum_{\mu} \eta_{\mu\mu} \left(N^{\mu} + \frac{1}{2} \eta^{\mu\mu} \right) \quad (5)$$

with creation/annihilation operators

$$a^{\mu} = \frac{1}{\sqrt{2}} (x^{\mu} + ip^{\mu}) \quad \bar{a}^{\mu} = \frac{1}{\sqrt{2}} (x^{\mu} - ip^{\mu}) \quad (6)$$

that satisfy

$$[a^{\mu}, \bar{a}^{\nu}] = \eta^{\mu\nu} \quad (7)$$

and mode number operators $N^{\mu} = \bar{a}^{\mu} a^{\mu}$ (no summation) that satisfy

$$[N^{\mu}, \bar{a}^{\nu}] = \eta^{\mu\nu} \bar{a}^{\mu} \quad [N^{\mu}, a^{\nu}] = -\eta^{\mu\nu} a^{\mu} \quad [N^{\mu}, N^{\nu}] = 0. \quad (8)$$

The products $|n\rangle = \prod_{\mu} |n^{\mu}\rangle$ of N^{μ} eigenstates form a Fock space of orthogonal oscillator modes with the ladder property

$$\bar{a}^{\mu} |n\rangle = e^{i\phi_{+}^{\mu}} \sqrt{n^{\mu} + \eta^{\mu\mu}} |n + \eta^{\mu\mu} \mathbf{e}_{\mu}\rangle \quad a^{\mu} |n\rangle = e^{i\phi_{-}^{\mu}} \sqrt{n^{\mu}} |n - \eta^{\mu\mu} \mathbf{e}_{\mu}\rangle, \quad (9)$$

where the \mathbf{e}_{μ} are unit vectors in the occupation number space

$$(\mathbf{e}_{\mu})^{\lambda} = \delta_{\mu}^{\lambda} \quad (10)$$

and the particular choice of phases $e^{i\phi_{+}^{\mu}}$ and $e^{i\phi_{-}^{\mu}}$, all taken to be 1 for the nonrelativistic Euclidean oscillator, has non-trivial consequences for the relativistic oscillator.

Kim and Noz [7] choose $e^{i\phi_+^\mu} = e^{i\phi_-^\mu} = 1$ for their relativistic oscillator model, so it follows from (9) that \bar{a}^0 acts as an annihilation operator and a^0 is the creation operator for the timelike mode, such that the ground state mode must have $n^0 \geq 1$. This role reversal between \bar{a}^0 and a^0 insures that timelike excitations have positive norm,

$$\langle n^0 \mathbf{n} | n^0 \mathbf{n} \rangle = \frac{1}{(n^0 - 1)!} \langle 1 \mathbf{0} | (\bar{a}^0)^{n^0-1} (a^0)^{n^0-1} | 1 \mathbf{0} \rangle = \langle 1 \mathbf{0} | (-\eta^{00})^{n^0-1} | 1 \mathbf{0} \rangle = 1 \quad (11)$$

but leads to an indefinite spectrum

$$\langle n^0 \mathbf{n} | K | n^0 \mathbf{n} \rangle = \omega \left(-n^0 + \sum_{\mu>0} n^\mu + \frac{1}{2} \eta_{\mu\nu} \eta^{\mu\nu} \right). \quad (12)$$

The requirements on the ground state

$$\bar{a}^0 |1 \mathbf{0}\rangle = 0 \quad a^\mu |1 \mathbf{0}\rangle = 0, \quad \mu > 0 \quad (13)$$

lead to a set of first order differential equations that reproduce the ground state solution proposed by Kim and Noz

$$\psi_0(x) = e^{-t^2} e^{-x^2/2} = e^{-(t^2 + \mathbf{x}^2)/2}. \quad (14)$$

In their study of quark dynamics, Feynman, Kislinger, and Ravndal [8] chose the phases

$$e^{i\phi_+^0} = e^{i\phi_-^0} = i \quad e^{i\phi_+^\mu} = e^{i\phi_-^\mu} = 1, \quad \mu > 0 \quad (15)$$

preserving the roles of \bar{a}^0 as creation operator and a^0 as annihilation operator for the timelike mode, under the requirement that $n^0 \leq 0$ so that

$$\bar{a}^0 |n\rangle = \sqrt{1 - n^0} |n - \mathbf{e}_0\rangle \quad a^0 |n\rangle = \sqrt{-n^0} |n + \mathbf{e}_0\rangle. \quad (16)$$

Although these states have positive spectrum

$$\langle -n^0 \mathbf{n} | K | -n^0 \mathbf{n} \rangle = \omega \left(n^0 + \sum_{\mu>0} n^\mu + \frac{1}{2} \eta_{\mu\nu} \eta^{\mu\nu} \right) \quad (17)$$

they have indefinite norm

$$\langle n^0 \mathbf{n} | n^0 \mathbf{n} \rangle = \frac{1}{(-n^0)!} \langle 0 | (a^0)^{-n^0} (\bar{a}^0)^{-n^0} | 0 \rangle = \langle 0 | (\eta^{00})^{-n^0} | 0 \rangle = (-1)^{-n^0} \quad (18)$$

requiring that the negative norm states (ghosts) be suppressed by exclusion of excited time-like modes. The first order differential equations $a^\mu \psi_0(x) = 0$ lead to the solution proposed in [8],

$$\psi_0(x) = e^{-x^2/2} = e^{-(\mathbf{x}^2 - t^2)/2} \quad (19)$$

with some regularization procedure required for normalization.

The ground state energy for uncoupled nonrelativistic oscillators can usually be found by associating $\frac{1}{2}\hbar\omega$ per degree of freedom. Although (12) and (17) indicate a ground state mass/energy in 4 dimensions of $2\hbar\omega$, the Horwitz-Arshansky solution exhibits the lower ground state level $\frac{3}{2}\hbar\omega$. Kastrup [1] has shown that the choice of Cartesian coordinates overlooks the singularity at the origin of polar coordinates, implicitly choosing one solution from a family of harmonic oscillators with different ground state levels. In the following sections we obtain solutions in polar coordinates for O(2), O(3), and the spacelike sector of O(2,1), and show that the ground state level depends on its angular eigenvalue.

2 Harmonic Oscillators in Polar Coordinates

In the polar coordinates appropriate to the oscillator problems in $D = 2$ and 3 dimensions

$$\begin{aligned} x &= \rho \cos \phi & y &= \rho \sin \phi & & \text{O}(2) \\ x &= \rho \cos \phi \sin \theta & y &= \rho \sin \phi \sin \theta & z &= \rho \cos \theta & \text{O}(3) \\ x &= \rho \cos \phi \cosh \beta & y &= \rho \sin \phi \cosh \beta & t &= \rho \sinh \beta & \text{O}(2,1) \end{aligned} \quad (20)$$

the Schrodinger equation takes the form

$$\left[-\partial_\rho^2 - \frac{D-1}{\rho} \partial_\rho + \frac{1}{\rho^2} \mathbf{M}^2 + \rho^2 - \varepsilon \right] \psi = 0 \quad (21)$$

where the energy/mass eigenvalue is $E = \frac{\omega}{2}\varepsilon$ and \mathbf{M}^2 is the Casimir operator of the symmetry group, formed from $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$, which we notate as

$$\begin{aligned} & & & M = x^1 p^2 - x^2 p^1 & & \text{O}(2) \\ L^1 = x^2 p^3 - x^3 p^2 & L^2 = x^3 p^1 - x^1 p^3 & M = x^1 p^2 - x^2 p^1 & & \text{O}(3) \\ A^1 = x^0 p^1 - x^1 p^0 & A^2 = x^0 p^2 - x^2 p^0 & M = x^1 p^2 - x^2 p^1 & & \text{O}(2,1) \end{aligned} \quad (22)$$

so that the parameterizations (20) diagonalize the M^{12} angular momentum component

$$M = x^1 p^2 - x^2 p^1 = -i\partial_\phi. \quad (23)$$

The Casimir operators in these coordinates are

$$\mathbf{M}^2 = \begin{cases} M^2 = -\partial_\phi^2 & \text{O(2)} \\ \mathbf{L}^2 = -\partial_\theta^2 - \frac{\cos \theta}{\sin \theta} \partial_\theta - \frac{1}{\sin^2 \theta} \partial_\phi^2 & \text{O(3)} \\ \mathbf{A}^2 = M^2 - \mathbf{A}^2 = \partial_\beta^2 + \frac{\sinh \beta}{\cosh \beta} \partial_\beta - \frac{1}{\cosh^2 \beta} \partial_\phi^2 & \text{O(2,1)} \end{cases} \quad (24)$$

so assuming a separation of variables

$$\begin{aligned} \psi(\rho, \phi) &= R(\rho) \Phi(\phi) & \text{O(2)} \\ \psi(\rho, \theta, \phi) &= R(\rho) F(\theta) \Phi(\phi) & \text{O(3)} \\ \psi(\rho, \beta, \phi) &= R(\rho) G(\beta) \Phi(\phi) & \text{O(2,1)} \end{aligned} \quad (25)$$

leads to the common angular function

$$\Phi(\phi) = e^{i\Lambda_1 \phi} \quad (26)$$

allowing the replacement of $M = -i\partial_\phi$ in (24) by its eigenvalue Λ_1 . For $D = 3$ a second separation of variables, associated with the eigenvalue equation $\mathbf{M}^2 \psi = \Lambda_2 \psi$ for the Casimir operators, leads to

$$(-\mathbf{M}^2 + \Lambda_2) F(\theta) = \left(\partial_\theta^2 + \frac{\cos \theta}{\sin \theta} \partial_\theta - \frac{\Lambda_1^2}{\sin^2 \theta} + \Lambda_2 \right) F(\theta) = 0 \quad (27)$$

$$(\mathbf{M}^2 - \Lambda_2) G(\beta) = \left(\partial_\beta^2 + \frac{\sinh \beta}{\cosh \beta} \partial_\beta + \frac{\Lambda_1^2}{\cosh^2 \beta} - \Lambda_2 \right) G(\beta) = 0 \quad (28)$$

which may be approached in two inequivalent ways. The first, following the method applied to the classical central force problems, notes the form of the first order derivative terms and substitutes

$$z = \cos \theta \quad \zeta = \sinh \beta \quad \Lambda_2 = l(l+1) \quad \Lambda_1 = m \quad (29)$$

so that the partial derivative terms for θ become

$$\partial_\theta = -\sin \theta \partial_z = -\sqrt{1-z^2} \partial_z \quad \frac{\cos \theta}{\sin \theta} \partial_\theta = -z \partial_z \quad (30)$$

$$\partial_\theta^2 = \sqrt{1-z^2} \partial_z \sqrt{1-z^2} \partial_z = (1-z^2) \partial_z^2 - z \partial_z \quad (31)$$

and for β become

$$\partial_\beta = \cosh \beta \partial_\zeta = \sqrt{1+\zeta^2} \partial_\zeta \quad \frac{\sinh \beta}{\cosh \beta} \partial_\beta = \zeta \partial_\zeta \quad (32)$$

$$\partial_\beta^2 = \sqrt{1+\zeta^2} \partial_\zeta \sqrt{1+\zeta^2} \partial_\zeta = (1+\zeta^2) \partial_\zeta^2 + \zeta \partial_\zeta. \quad (33)$$

Writing $F(\theta) \rightarrow P_l^m(z)$ and $G(\beta) \rightarrow \hat{P}_l^m(\zeta)$ equations (27) and (28) become solutions to the associated Legendre equation in the respective forms

$$\left[(1 - z^2) \partial_z^2 - 2z \partial_z + l(l+1) - \frac{m^2}{1 - z^2} \right] P_l^m(z) = 0 \quad (34)$$

$$\left[(1 + \zeta^2) \partial_\zeta^2 + 2\zeta \partial_\zeta - l(l+1) + \frac{m^2}{1 + \zeta^2} \right] \hat{P}_l^m(\zeta) = 0. \quad (35)$$

Notice that (35) can be obtained from (34) by letting

$$z = i\zeta \rightarrow z^2 = -\zeta^2 \quad \partial_z^2 \rightarrow -\partial_\zeta^2 \quad z \partial_z \rightarrow \zeta \partial_\zeta. \quad (36)$$

A second, qualitatively different set of solutions is obtained by substituting

$$z = \frac{\cos \theta}{\sin \theta} \quad F_l^m(z) = (1 + z^2)^{\frac{1}{4}} \hat{P}_l^m(z) \quad (37)$$

$$\zeta = \frac{\sinh \beta}{\cosh \beta} \quad G_l^m(\zeta) = (1 - \zeta^2)^{\frac{1}{4}} P_l^m(\zeta) \quad (38)$$

so that the partial derivative terms for θ become

$$\partial_\theta = -\frac{1}{\sin^2 \theta} \partial_z = -(1 + z^2) \partial_z \quad \frac{\cos \theta}{\sin \theta} \partial_\theta = -z(1 + z^2) \partial_z \quad (39)$$

$$\partial_\theta^2 = (1 + z^2) \partial_z (1 + z^2) \partial_z = (1 + z^2)^2 \partial_z^2 + (1 + z^2) 2z \partial_z \quad (40)$$

$$\partial_\theta^2 + \cot \theta \partial_\theta = (1 + z^2) [(1 + z^2) \partial_z^2 + z \partial_z] \quad (41)$$

and for β become

$$\partial_\beta = \frac{1}{\cosh^2 \beta} \partial_\zeta = (1 - \zeta^2) \partial_\zeta \quad \frac{\sinh \beta}{\cosh \beta} \partial_\beta = \zeta (1 - \zeta^2) \partial_\zeta \quad (42)$$

$$\partial_\beta^2 = (1 - \zeta^2) \partial_\zeta (1 - \zeta^2) \partial_\zeta = (1 - \zeta^2) [(1 - \zeta^2) \partial_\zeta^2 - 2\zeta \partial_\zeta] \quad (43)$$

$$\partial_\beta^2 + \tanh \beta \partial_\beta = (1 - \zeta^2) [(1 - \zeta^2) \partial_\zeta^2 - \zeta \partial_\zeta]. \quad (44)$$

Using

$$\partial_z (1 + z^2)^{\frac{1}{4}} \hat{P}(z) = (1 + z^2)^{\frac{1}{4}} \left[\partial_z + \frac{1}{2} \frac{z}{1 + z^2} \right] \hat{P}(z) \quad (45)$$

$$\partial_\zeta (1 - \zeta^2)^{\frac{1}{4}} P(\zeta) = (1 - \zeta^2)^{\frac{1}{4}} \left[\partial_\zeta - \frac{1}{2} \frac{\zeta}{1 - \zeta^2} \right] P(\zeta) \quad (46)$$

we are led to associated Legendre equations

$$\left[(1+z^2) \partial_z^2 + 2z \partial_z - m(m+1) + \frac{l^2}{1+z^2} \right] \hat{P}_m^l(z) = 0 \quad (47)$$

$$\left[(1-\zeta^2) \partial_\zeta^2 - 2\zeta \partial_\zeta + m(m+1) - \frac{l^2}{1-\zeta^2} \right] P_m^l(\zeta) = 0 \quad (48)$$

where the constants l and m have reversed roles with respect to equations (34) and (35), having been introduced to satisfy

$$m(m+1) = \Lambda_1^2 - \frac{1}{4} \quad \longrightarrow \quad \Lambda_1 = m + \frac{1}{2} \quad (49)$$

$$l^2 = \Lambda_2^2 + \frac{1}{4} \quad \longrightarrow \quad \Lambda_2 = l^2 - \frac{1}{4}. \quad (50)$$

Comparing (29) and (49), we write $\Lambda_1 = m + s$, so that (26) becomes

$$\Phi(\phi) = e^{i\Lambda_1\phi} = e^{i(m+s)\phi} \quad (51)$$

where, for $D = 3$, the orbital angular momentum is characterized by $s = 0, 1/2$ and may be integral or half-integral. For $D = 2$ we assume that s can be any real constant.

The remaining radial equations are

$$\left[-\partial_\rho^2 - \frac{1}{\rho} \partial_\rho + \frac{1}{\rho^2} \Lambda_1^2 + \rho^2 - \varepsilon \right] R(\rho) = 0, \quad D = 2 \quad (52)$$

$$\left[-\partial_\rho^2 - \frac{2}{\rho} \partial_\rho + \frac{1}{\rho^2} \Lambda_2^2 + \rho^2 - \varepsilon \right] R(\rho) = 0, \quad D = 3. \quad (53)$$

The change of variables

$$x = \rho^2 \quad (54)$$

which entails

$$\partial_r = 2x^{\frac{1}{2}} \partial_x \quad \partial_r^2 = 2x^{\frac{1}{2}} \partial_x 2x^{\frac{1}{2}} \partial_x = 4x \partial_x^2 + 2 \partial_x, \quad (55)$$

and the substitutions

$$R(\rho) = x^{(m+s)/2} e^{-x/2} L(x) \quad D = 2 \quad (56)$$

$$R(\rho) = x^{(m-s)/2} e^{-x/2} L(x) \quad D = 3 \quad (57)$$

in radial equations (52) and (53) lead to Laguerre equations, where in $D = 2$, $L(x)$ satisfies

$$\left[x \partial_x^2 + (m+s-x+1) \partial_x + \frac{1}{2} \left(\frac{1}{2} \varepsilon - m - s - 1 \right) \right] L_n^\alpha(x) = 0 \quad (58)$$

$$\alpha = m + s \quad n = \frac{1}{2} \left(\frac{1}{2} \varepsilon - m - s - 1 \right) \quad (59)$$

and in $D = 3$, $L(x)$ satisfies

$$\left[x \partial_x^2 + \left(l - s - x + \frac{3}{2} \right) \partial_x + \frac{1}{2} \left(\frac{1}{2} \varepsilon - l + s - \frac{3}{2} \right) \right] L_n^\alpha(x) = 0 \quad (60)$$

$$\alpha = l - s + \frac{1}{2} \quad n = \frac{1}{2} \left(\frac{1}{2} \varepsilon - l + s - \frac{3}{2} \right). \quad (61)$$

From $E = \frac{\omega}{2} \varepsilon$ the spectra are given by

$$E = \omega (2n + m + s + 1) \quad D = 2 \quad (62)$$

$$E = \omega (2n + l + 3/2 - s) \quad D = 3 \quad (63)$$

and the wavefunctions are

$$\psi_{nm}^{O(2),s}(\rho, \phi) = A_{nm} e^{-\rho^2/2} \rho^{m+s} L_n^{m+s}(\rho^2) e^{i(m+s)\phi} \quad (64)$$

$$\psi_{nlm}^{O(3),s=0}(\rho, \theta, \phi) = A_{nlm} e^{-\rho^2/2} \rho^l L_n^{l+\frac{1}{2}}(\rho^2) P_l^m(\cos \theta) e^{im\phi} \quad (65)$$

$$\psi_{nlm}^{O(3),s=\frac{1}{2}}(\rho, \theta, \phi) = A_{nlm} e^{-\rho^2/2} \rho^{l-\frac{1}{2}} L_n^l(\rho^2) \frac{\hat{P}_m^l(\cot \theta)}{\sqrt{|\sin \theta|}} e^{i(m+\frac{1}{2})\phi} \quad (66)$$

$$\psi_{nlm}^{O(2,1),s=0}(\rho, \beta, \phi) = A_{nlm} e^{-\rho^2/2} \rho^l L_n^{l+\frac{1}{2}}(\rho^2) \hat{P}_l^m(\sinh \beta) e^{im\phi} \quad (67)$$

$$\psi_{nlm}^{O(2,1),s=\frac{1}{2}}(\rho, \beta, \phi) = A_{nlm} e^{-\rho^2/2} \rho^{l-\frac{1}{2}} L_n^l(\rho^2) \frac{P_m^l(\tanh \beta)}{\sqrt{\cosh \beta}} e^{i(m+\frac{1}{2})\phi}. \quad (68)$$

Using the properties $L_\beta^\alpha = 0$ for $\beta < 0$ and $L_0^\alpha = P_0^0 = 1$, the wavefunctions with eigenvalues $n = l = m = 0$ are summarized as

$$\psi_0^{O(2),s}(\rho, \phi) = A_0 e^{-\rho^2/2} (\rho e^{i\phi})^s = A_0 e^{-(x^2+y^2)/2} (x^2 + y^2)^{s/2} e^{is \arctan(\frac{y}{x})} \quad (69)$$

$$\psi_0^{O(3),s}(\rho, \theta, \phi) = A_0 e^{-\rho^2/2} \frac{e^{is\phi}}{(\rho |\sin \theta|)^s} = A_0 e^{-(x^2+y^2+z^2)/2} \frac{e^{is \arctan(\frac{y}{x})}}{(x^2 + y^2)^{s/2}} \quad (70)$$

$$\psi_0^{O(2,1),s}(\rho, \beta, \phi) = A_0 e^{-\rho^2/2} \frac{e^{is\phi}}{(\rho \cosh \beta)^s} = A_0 e^{-(x^2+y^2-t^2)/2} \frac{e^{is \arctan(\frac{y}{x})}}{(x^2 + y^2)^{s/2}} \quad (71)$$

and so, as expected, are separable in Cartesian coordinates only for $s = 0$, in which case they recover the standard solutions expressed as products of one dimensional oscillators. In particular, the $s = 0$ ground state for $O(2,1)$ is precisely the state proposed by Feynman, Kislinger, and Ravndal.

3 Number Representation in Polar Coordinates

A number representation appropriate to the solutions (64) to (68) consists of polar creation/annihilation operators that act on polar eigenstates of the total mode number N and the symmetry operators \mathbf{M}^2 and M to produce new polar eigenstates. The resulting representation will be equivalent to the standard Cartesian Fock space if the polar eigenstates are unitarily connected to the Cartesian number states, in which case they can be found by expressing N, \mathbf{M}^2 and M in terms of \bar{a}^μ and a^μ and diagonalizing the resulting operators. We consider the Cartesian multiplet φ_1 of first excited states as arising from the action of the vector multiplet of creation operators on the ground state φ_0 . Thus, in $D = 2$ the vector operator multiplet takes φ_0 to φ_1 as

$$\varphi_1 = \begin{pmatrix} \varphi_{10} \\ \varphi_{01} \end{pmatrix} = \begin{pmatrix} \bar{a}^1 \varphi_0 \\ \bar{a}^2 \varphi_0 \end{pmatrix} = \begin{pmatrix} \bar{a}^1 \\ \bar{a}^2 \end{pmatrix} \varphi_0. \quad (72)$$

Using (6) to replace x^μ and p^μ with \bar{a}^μ and a^μ , the angular momentum operator

$$M = x^1 p^2 - x^2 p^1 = -i (\bar{a}^1 a^2 - \bar{a}^2 a^1) \quad (73)$$

is seen to act on φ_1 as

$$M \varphi_1 = -i (\bar{a}^1 a^2 - \bar{a}^2 a^1) \begin{pmatrix} \bar{a}^1 \varphi_0 \\ \bar{a}^2 \varphi_0 \end{pmatrix} = \begin{pmatrix} i \bar{a}^2 \varphi_0 \\ -i \bar{a}^1 \varphi_0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \varphi_1 \quad (74)$$

and so has eigenvalues ± 1 on eigenstates

$$\tilde{\varphi}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_{10} + i \varphi_{01} \\ -\varphi_{10} + i \varphi_{01} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{a}_+ \\ -\bar{a}_- \end{pmatrix} \varphi_0 \quad (75)$$

where the polar creation/annihilation operators

$$a_\pm = \frac{1}{\sqrt{2}} (a^1 \pm i a^2) \quad \bar{a}_\pm = \frac{1}{\sqrt{2}} (\bar{a}^1 \pm i \bar{a}^2) \quad (76)$$

commute among themselves except for

$$[a_+, \bar{a}_-] = [a_-, \bar{a}_+] = 1. \quad (77)$$

Since a^1 and a^2 commute with \bar{a}^0 and \bar{a}^3 , the operators defined in (76) similarly diagonalize M in $D = 3$, with eigenvalue 0 on the states $\bar{a}^0 \varphi_0$ and $\bar{a}^3 \varphi_0$.

3.1 Number representation for $D = 2$

Because $\mathbf{M}^2 = (M)^2$ in $D = 2$, operators (76) are sufficient to fully characterize the $O(2)$ oscillator. From the four available products

$$\bar{a}_+ a_+ = \frac{1}{2} (\bar{a}^1 + i\bar{a}^2) (a^1 + ia^2) = \frac{1}{2} (N^1 - N^2 + i\bar{a}^2 a^1 + i\bar{a}^1 a^2) \quad (78)$$

$$\bar{a}_- a_- = \frac{1}{2} (\bar{a}^1 - i\bar{a}^2) (a^1 - ia^2) = \frac{1}{2} (N^1 - N^2 - i\bar{a}^2 a^1 - i\bar{a}^1 a^2) \quad (79)$$

$$\bar{a}_+ a_- = \frac{1}{2} (\bar{a}^1 + i\bar{a}^2) (a^1 - ia^2) = \frac{1}{2} (N^1 + N^2 + i\bar{a}^2 a^1 - i\bar{a}^1 a^2) \quad (80)$$

$$\bar{a}_- a_+ = \frac{1}{2} (\bar{a}^1 - i\bar{a}^2) (a^1 + ia^2) = \frac{1}{2} (N^1 + N^2 - i\bar{a}^2 a^1 + i\bar{a}^1 a^2) \quad (81)$$

we may form the symmetric Hermitian combinations

$$N = \bar{a}_+ a_- + \bar{a}_- a_+ = N^1 + N^2 \quad (82)$$

$$\Delta = \bar{a}_+ a_+ - \bar{a}_- a_- = N^1 - N^2 = \bar{a}^1 a^1 - \bar{a}^2 a^2 \quad (83)$$

and the antisymmetric Hermitian combinations

$$M = \bar{a}_+ a_- - \bar{a}_- a_+ = -i (\bar{a}^1 a^2 - \bar{a}^2 a^1) \quad (84)$$

$$Q = -i (\bar{a}_+ a_+ - \bar{a}_- a_-) = \bar{a}^1 a^2 + \bar{a}^2 a^1. \quad (85)$$

In Cartesian coordinates, the maximal set of commuting operators is $\{N^1, N^2\}$, and from these we construct the Hamiltonian. Using

$$[M, N^1] = -i [\bar{a}^1 a^2 - \bar{a}^2 a^1, \bar{a}^1 a^1] = i (\bar{a}^1 a^2 + \bar{a}^2 a^1) = iQ \quad (86)$$

$$[M, N^2] = -i [\bar{a}^1 a^2 - \bar{a}^2 a^1, \bar{a}^2 a^2] = -i (\bar{a}^1 a^2 + \bar{a}^2 a^1) - iQ \quad (87)$$

we confirm that angular momentum commutes with the total mode number

$$[M, N] = [M, N^1] + [M, N^2] = 0 \quad (88)$$

but since N^1 and N^2 do not commute with M , they are not separately observable in the polar representation.

Since N is a positive operator, we must address the problem of negative energy states. From (62) the energy of $n = 0$ states can become negative if $m + s < -1$. For eigenvalues $n \geq 0$

and $m + s \geq 0$, the wavefunctions (64) are made orthonormal by taking the normalization to be

$$A_{nm} = \frac{(-1)^n}{\sqrt{\int_0^{2\pi} d\phi \int_0^\infty \rho d\rho \left| \psi_{nm}^{O(2),s}(\rho, \phi) \right|^2}} = (-1)^n \sqrt{\frac{\Gamma(n+1)}{\pi \Gamma(n+m+s+1)}} \quad (89)$$

but for states with $n = 0$ and $m + s < 0$, this becomes

$$A_{0m} = \frac{1}{\sqrt{\int_0^{2\pi} d\phi \int_0^\infty \rho d\rho \left| \psi_{0m}^{O(2),s}(\rho, \phi) \right|^2}} = \frac{1}{\sqrt{\pi \int_0^\infty dx e^{-x/2} x^{m+s}}} \xrightarrow{m+s < 0} 0 \quad (90)$$

eliminating negative energy states ψ_{0m} for $m + s < 0$. The general normalized wavefunction is then

$$\psi_{nm}(r, \phi) = \begin{cases} (-1)^n \sqrt{\frac{\Gamma(n+1)}{\pi \Gamma(n+m+s+1)}} r^{m+s} e^{-\frac{r^2}{2}} L_n^{m+s}(r^2) e^{i(m+s)\phi} & , n \geq 0, n+m+s \geq 0 \\ 0 & , \text{otherwise} \end{cases} \quad (91)$$

with positive definite total energy. We may satisfy the requirement

$$n + m + s \geq 0 \quad (92)$$

by taking $0 \leq s < 1$ and $m \geq 0$. From (5) the Hamiltonian in $D = 2$ is $H = \omega(N + 1)$ so comparing with (62) we find

$$N = 2n + m + s \quad (93)$$

where to avoid confusion with the principal quantum number n , we use N to represent both the total mode number operator and its eigenvalue. Using the commutation relations

$$[N, a_\pm] = -a_\pm \quad [N, \bar{a}_\pm] = \bar{a}_\pm \quad (94)$$

$$[M, a_\pm] = \pm a_\pm \quad [M, \bar{a}_\pm] = \pm \bar{a}_\pm. \quad (95)$$

we compare

$$Na_+ \psi_{nm} = a_+ (N - 1) \psi_{nm} = (2n + m + s - 1) a_+ \psi_{nm} \quad (96)$$

$$Ma_+ \psi_{nm} = (m + s + 1) a_+ \psi_{nm} \quad (97)$$

with

$$N\psi_{n-1,m+1} = (2n + m + s - 1) \psi_{n-1,m+1} \quad (98)$$

$$M\psi_{n-1,m+1} = (m + s + 1) \psi_{n-1,m+1} \quad (99)$$

and conclude that

$$a_+ \psi_{nm} = C_{nm}^+ \psi_{n-1,m+1} \quad (100)$$

where C_{nm}^+ is a complex coefficient, with norm found from

$$\|a_+ \psi_{nm}\|^2 = \langle \psi_{nm} | \bar{a}_- a_+ | \psi_{nm} \rangle = |C_{nm}^+|^2 \langle \psi_{n-1,m+1} | \psi_{n-1,m+1} \rangle = |C_{nm}^+|^2 \quad (101)$$

Similarly comparing,

$$N a_- \psi_{nm} = (2n + m + s - 1) a_- \psi_{nm} \quad (102)$$

$$M a_- \psi_{nm} = (m + s - 1) a_- \psi_{nm} \quad (103)$$

$$N \bar{a}_+ \psi_{nm} = (2n + m + s + 1) \bar{a}_+ \psi_{nm} \quad (104)$$

$$M \bar{a}_+ \psi_{nm} = (m + s + 1) \bar{a}_+ \psi_{nm} \quad (105)$$

$$N \bar{a}_- \psi_{nm} = (2n + m + s + 1) \bar{a}_- \psi_{nm} \quad (106)$$

$$M \bar{a}_- \psi_{nm} = (m + s - 1) \bar{a}_- \psi_{nm} \quad (107)$$

with

$$N \psi_{n,m-1} = (2n + (m - 1)) \psi_{n,m-1} = (2n + s + m - 1) \psi_{n,m-1} \quad (108)$$

$$M \psi_{n,m-1} = (m - 1) \psi_{n,m-1} \quad (109)$$

$$N \psi_{n,m+1} = [2n + s + (m + 1)] \psi_{n,m+1} = (2n + s + m + 1) \psi_{n,m+1} \quad (110)$$

$$M \psi_{n,m+1} = (m + s + 1) \psi_{n,m+1} \quad (111)$$

$$N \psi_{n+1,m-1} = [2(n + 1) + s + (m - 1)] \psi_{n+1,m-1} = (2n + s + m + 1) \psi_{n+1,m-1} \quad (112)$$

$$M \psi_{n+1,m-1} = (m + s - 1) \psi_{n+1,m-1} \quad (113)$$

leads to

$$a_- \psi_{nm} = C_{nm}^- \psi_{nm-1} \quad \bar{a}_+ \psi_{nm} = \bar{C}_{nm}^+ \psi_{nm+1} \quad \bar{a}_- \psi_{nm} = \bar{C}_{nm}^- \psi_{n+1,m-1}. \quad (114)$$

We eliminate two coefficients

$$|\bar{C}_{nm}^-|^2 = |C_{nm}^+|^2 + 1 \quad |\bar{C}_{nm}^+|^2 = |C_{nm}^-|^2 + 1 \quad (115)$$

using the commutation relations (77). Solving

$$2n + m + s = \langle nm | N | nm \rangle = \langle nm | \bar{a}_+ a_- + \bar{a}_- a_+ | nm \rangle = |C_{nm}^-|^2 + |C_{nm}^+|^2 \geq 0, \quad (116)$$

which requires that the total mode number be positive, together with

$$m + s = \langle nm | M | nm \rangle = \langle nm | \bar{a}_+ a_- - \bar{a}_- a_+ | nm \rangle = |C_{nm}^-|^2 - |C_{nm}^+|^2 \quad (117)$$

and taking the coefficients to be real

$$C_{nm}^+ = \sqrt{n} \quad \bar{C}_{nm}^- = \sqrt{n+1} \quad (118)$$

$$C_{nm}^- = \sqrt{n+m+s} \quad \bar{C}_{nm}^+ = \sqrt{n+m+s+1} \quad (119)$$

we write the actions of the ladder operators as

$$a_+ \psi_{nm} = \sqrt{n} \psi_{n-1, m+1} \quad \bar{a}_- \psi_{nm} = \sqrt{n+1} \psi_{n+1, m-1} \quad (120)$$

$$a_- \psi_{nm} = \sqrt{n+m+s} \psi_{n, m-1} \quad \bar{a}_+ \psi_{nm} = \sqrt{n+m+s+1} \psi_{n, m+1} . \quad (121)$$

Special care must be taken with the ground state (69) because (120) and (121) lead to

$$a_+ \psi_0 = 0 \quad a_- \psi_0 = \sqrt{s} \psi_{0, -1} \quad (122)$$

or equivalently

$$a^1 \psi_0 = \sqrt{\frac{s}{2}} \psi_{0, -1} \quad a^2 \psi_0 = i \sqrt{\frac{s}{2}} \psi_{0, -1} \quad (123)$$

suggesting a negative energy state. However, the well-defined, non-zero function $\psi_{0, -1}$ is non-normalizable and by (91) does not correspond to any state in the Fock space. We interpret the action of a_- in (122) as taking the ground state to a non-observable function which must be taken account in calculations such as

$$N \psi_0 = (\bar{a}_+ a_- + \bar{a}_- a_+) \psi_0 = \sqrt{s} \bar{a}_+ \psi_{0, -1} = \sqrt{s} \sqrt{-1 + s + 1} \psi_0 = s \psi_0 \quad (124)$$

but is effectively annihilated at the end of calculations.

We may construct excited states from the ground state as

$$\zeta_{\alpha\beta} = \frac{1}{N_{\alpha\beta}} (\bar{a}_+)^{\alpha} (\bar{a}_-)^{\beta} \psi_0 \quad (125)$$

with normalization coefficient $N_{\alpha\beta}$. It follows from (120) that

$$(\bar{a}_-)^{\beta} \psi_0 = \sqrt{\beta!} \psi_{\beta,-\beta} \quad (126)$$

and from (121) that

$$(\bar{a}_+)^{\alpha} \psi_{\beta,-\beta} = \sqrt{\frac{\Gamma(s+\alpha+1)}{\Gamma(s+1)}} \psi_{\beta,-\beta+\alpha} \quad (127)$$

and so we take

$$N_{\alpha\beta} = \sqrt{\beta! \frac{\Gamma(s+\alpha+1)}{\Gamma(s+1)}} \quad (128)$$

which reduces to $\sqrt{\alpha!\beta!}$ in the case $s = 0$. Operating on these states with the total mode operator (82)

$$N\zeta_{\alpha\beta} = \frac{1}{N_{\alpha\beta}} (\bar{a}_+ a_- + \bar{a}_- a_+) (\bar{a}_+)^{\alpha} (\bar{a}_-)^{\beta} \psi_0 \quad (129)$$

with the commutation relations (77) and the identity

$$[B, A] = c \longrightarrow [B, A^n] = cnA^{n-1} \quad (130)$$

we calculate

$$\bar{a}_+ a_- (\bar{a}_+)^{\alpha} (\bar{a}_-)^{\beta} \psi_0 = \bar{a}_+ [(\bar{a}_+)^{\alpha} a_- + \alpha (\bar{a}_+)^{\alpha-1}] (\bar{a}_-)^{\beta} \psi_0 \quad (131)$$

$$= [(\bar{a}_+)^{\alpha+1} (\bar{a}_-)^{\beta} a_- + \alpha (\bar{a}_+)^{\alpha} (\bar{a}_-)^{\beta}] \psi_0 \quad (132)$$

$$= (\bar{a}_+)^{\alpha} (\bar{a}_-)^{\beta} (\bar{a}_+ a_- + \alpha) \psi_0 \quad (133)$$

$$\bar{a}_- a_+ (\bar{a}_+)^{\alpha} (\bar{a}_-)^{\beta} \psi_0 = (\bar{a}_+)^{\alpha} \bar{a}_- a_+ (\bar{a}_-)^{\beta} \psi_0 \quad (134)$$

$$= (\bar{a}_+)^{\alpha} \bar{a}_- [(\bar{a}_-)^{\beta} a_+ + \beta (\bar{a}_-)^{\beta-1}] \psi_0 \quad (135)$$

$$= (\bar{a}_+)^{\alpha} (\bar{a}_-)^{\beta} (\bar{a}_- a_+ + \beta) \psi_0 \quad (136)$$

which combine to

$$N\zeta_{\alpha\beta} = \frac{1}{N_{\alpha\beta}} (\bar{a}_+)^{\alpha} (\bar{a}_-)^{\beta} (\alpha + \beta + \bar{a}_+ a_- + \bar{a}_- a_+) \psi_0 \quad (137)$$

and using (124) we show that

$$N\zeta_{\alpha\beta} = (\alpha + \beta + s) \zeta_{\alpha\beta} \quad (138)$$

so that the states $\zeta_{\alpha\beta}$ have total mode number given by

$$N = \alpha + \beta + s = \alpha + \beta + N_{\text{ground state}} . \quad (139)$$

A similar calculation using (84) leads to

$$M\zeta_{\alpha\beta} = (\alpha - \beta + s)\zeta_{\alpha\beta} \quad (140)$$

so that the states $\zeta_{\alpha\beta}$ have angular momentum

$$M = \alpha - \beta + s = \alpha - \beta + M_{\text{ground state}} \quad m = \alpha - \beta . \quad (141)$$

Comparing (138) with (93) we see that

$$\alpha + \beta = 2n + m \quad (142)$$

which combines with (141) and (139) to provide expressions for the principal quantum number n and the integer part of the angular momentum m

$$n = \frac{1}{2}(\alpha + \beta - m) = \beta \quad m = \alpha - \beta = N - s - 2\beta \quad (143)$$

and fixes α as

$$\alpha = 2n + m - \beta = n + m = N - s - m. \quad (144)$$

Thus, the states $\zeta_{\alpha\beta}$ defined in (125) can be identified with explicit solutions $\psi_{n,m}^{(N)}$ through

$$\zeta_{\alpha\beta} = \psi_{\beta, \alpha-\beta}^{(\alpha+\beta+s)} \quad \psi_{n,m}^{(N)} = \zeta_{N-s-m,n} \quad (145)$$

for which $n = \beta = 0, 1, \dots, N - s$ characterizes the $(N - s + 1)$ -fold multiplicity of states with mode number N . Equivalently, the multiplicity can be enumerated by the angular momentum m , and for $s = 0$, this simple multiplicity structure is identical to the Cartesian picture in which there are $N + 1$ ways to build a state of total mode number N from a pair of one dimensional oscillators.

Bars has recently observed [2] that the harmonic oscillator Hamiltonian in D dimensions possesses a symmetry generated by the products $\bar{a}^\mu a^\nu$ of the ladder operators. Because such products replace one ν -mode of the oscillator with one μ -mode, the total mode number, and therefore the total mass/energy, is conserved. The traceless part of the generators

$$J^{\mu\nu} = \bar{a}^\mu a^\nu - \frac{1}{D}\eta^{\mu\nu}\eta_{\lambda\rho}\bar{a}^\lambda a^\rho \quad (146)$$

generates an $SU(D-1, 1)$ or $SU(D)$ dynamical symmetry of the Hamiltonian, while the trace

$$\eta_{\lambda\rho}\bar{a}^\lambda a^\rho = \sum_\mu \eta_{\mu\mu} N^\mu = N \quad (147)$$

is the total mode number and differs from the Hamiltonian by a c-number. The antisymmetric part of the generators

$$\frac{1}{2}(J^{\mu\nu} - J^{\nu\mu}) = M^{\mu\nu} = \bar{a}^\mu a^\nu - \bar{a}^\nu a^\mu \quad (148)$$

generates the $SO(D-1, 1)$ or $SO(D)$ symmetry of the Hamiltonian. Bars argues that harmonic oscillator states should belong to representations of the SU dynamical symmetry as well as to representations of the SO symmetry, imposing additional constraints on admissible solutions.

For the Cartesian ladder operators in two dimensions, the traceless operator is

$$J = \begin{bmatrix} \frac{1}{2}(\bar{a}^1 a^1 - \bar{a}^2 a^2) & \bar{a}^1 a^2 \\ \bar{a}^2 a^1 & -\frac{1}{2}(\bar{a}^1 a^1 - \bar{a}^2 a^2) \end{bmatrix} \quad (149)$$

with antisymmetric part equal to the angular momentum operator

$$M^{12} = \bar{a}^1 a^2 - \bar{a}^2 a^1 = iM \quad (150)$$

and symmetric part given by

$$S = \frac{1}{2}(J + J^\dagger) = \frac{1}{2} \begin{bmatrix} \bar{a}^1 a^1 - \bar{a}^2 a^2 & \bar{a}^1 a^2 + \bar{a}^2 a^1 \\ \bar{a}^1 a^2 + \bar{a}^2 a^1 & -\bar{a}^1 a^1 + \bar{a}^2 a^2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \Delta & iQ \\ iQ & -\Delta \end{bmatrix} \quad (151)$$

where we use (83) and (85). Directly calculating

$$\left[\frac{1}{2}M, \frac{1}{2}\Delta\right] = \frac{1}{4}[M, N^1] - \frac{1}{4}[M, N^2] = i\frac{1}{2}Q \quad (152)$$

$$\left[\frac{1}{2}\Delta, \frac{1}{2}Q\right] = -i\frac{1}{4}[\bar{a}_+ a_+ + \bar{a}_- a_-, \bar{a}_+ a_+ - \bar{a}_- a_-] = \frac{1}{2}i[\bar{a}_+ a_+, \bar{a}_- a_-] = i\frac{1}{2}M \quad (153)$$

$$\left[\frac{1}{2}Q, \frac{1}{2}M\right] = -i\frac{1}{4}[\bar{a}^1 a^2 + \bar{a}^2 a^1, \bar{a}^1 a^2 - \bar{a}^2 a^1] = \frac{1}{2}i[\bar{a}^1 a^2, \bar{a}^2 a^1] = i\frac{1}{2}\Delta \quad (154)$$

we verify that the three independent operators $\{\frac{1}{2}M, \frac{1}{2}\Delta, \frac{1}{2}Q\}$ satisfy the $SU(2)$ algebra. Equation (88) confirms that M commutes with total mode number N — similarly,

$$[N, \Delta] = [N^1 + N^2, N^1 - N^2] = 0 \quad (155)$$

$$[N, Q] = [\bar{a}^1 a^1 + \bar{a}^2 a^2, \bar{a}^1 a^2 + \bar{a}^2 a^1] = 0 \quad (156)$$

so this $SU(2)$ is indeed a symmetry of the Hamiltonian. In Cartesian coordinates, the operator Δ is chosen to be observable, while in polar coordinates the operator M is observable. Comparison of (123) and (151) however indicates that the $SU(2)$ symmetry is spontaneously broken for the states (91) except in the case that $s = 0$. Moreover, expanding the $SU(2)$ Casimir operator in terms of the creation and annihilation operators, (83) — (85) lead to

$$\left(\frac{1}{2}M\right)^2 + \left(\frac{1}{2}\Delta\right)^2 + \left(\frac{1}{2}Q\right)^2 = \frac{1}{4}N(N+2) = \frac{1}{2}N\left(\frac{1}{2}N+1\right). \quad (157)$$

Since the Casimir eigenvalue $\frac{1}{2}N$ of a unitary representation of $SU(2)$ must be integral or half-integral, the $s \neq 0$ solutions appear to violate unitarity [3]. A more detailed study of the unitarity of the explicit solutions will be presented in a subsequent paper.

To verify that the solutions (91) form the basis for a representation of the operator algebra, we express the creation/annihilation operators in polar coordinates. Combining (6) and (76) as

$$a_{\pm} = \frac{1}{2}[(x + \partial_x) \pm i(y + \partial_y)] = \frac{1}{2}[x \pm iy + (\partial_x \pm i\partial_y)] \quad (158)$$

$$\bar{a}_{\pm} = \frac{1}{2}[(x - \partial_x) \pm i(y - \partial_y)] = \frac{1}{2}[x \pm iy - (\partial_x \pm i\partial_y)] \quad (159)$$

we obtain the polar expressions

$$a_{\pm} = \frac{1}{2}e^{\pm i\phi} \left(\rho + \frac{\partial}{\partial \rho} \pm \frac{i}{\rho} \frac{\partial}{\partial \phi} \right) \quad \bar{a}_{\pm} = \frac{1}{2}e^{\pm i\phi} \left[\rho - \left(\frac{\partial}{\partial \rho} \pm \frac{i}{\rho} \frac{\partial}{\partial \phi} \right) \right]. \quad (160)$$

Applying the annihilation operators to the ground state (69), we recover (122) in the explicit form

$$a_+ \psi_0 = \frac{1}{2} \sqrt{\frac{1}{\pi \Gamma(s+1)}} \left(\rho + (s - \rho^2) \frac{1}{\rho} - \frac{1}{\rho} s \right) e^{-\rho^2/2} \rho^s e^{i\phi(s+1)} = 0 \quad (161)$$

$$a_- \psi_0 = \sqrt{s} \sqrt{\frac{1}{\pi \Gamma(s)}} e^{-\rho^2/2} (\rho e^{i\phi})^{s-1}, \quad (162)$$

where the result in (162) is formally equivalent to $\sqrt{s}\psi_{0,-1}$ but as discussed above, does not correspond to any state in the Fock space, and we treat as annihilation. For the general state (91), using the notation of (54) $x = \rho^2$, the ρ derivative is

$$\frac{\partial}{\partial \rho} \left[e^{-\rho^2/2} L_n^{m+s}(\rho^2) (\rho e^{i\phi})^{m+s} \right] = \frac{e^{-\rho^2/2} (\rho e^{i\phi})^{m+s}}{\rho} \left[-\rho^2 + m + s + 2x \frac{d}{dx} \right] L_n^{m+s}(x) \quad (163)$$

providing

$$\frac{\partial}{\partial \rho} \psi_{nm} = \left(-\rho + \frac{m+s}{\rho} \right) \psi_{nm} + \frac{2}{\rho} A_{nm} e^{-\rho^2/2} (\rho e^{i\phi})^{m+s} x \frac{d}{dx} L_n^{m+s}(x) \quad (164)$$

and the ϕ derivative is

$$\frac{i}{\rho} \frac{\partial}{\partial \phi} \psi_{nm} = -\frac{m+s}{\rho} \psi_{nm} \quad (165)$$

so that

$$\left(\frac{\partial}{\partial \rho} + \frac{i}{\rho} \frac{\partial}{\partial \phi} \right) \psi_{nm} = -\rho \psi_{nm} + \frac{2}{\rho} A_{nm} e^{-\rho^2/2} (\rho e^{i\phi})^{m+s} x \frac{d}{dx} L_n^{m+s}(x) \quad (166)$$

$$\begin{aligned} \left(\frac{\partial}{\partial \rho} - \frac{i}{\rho} \frac{\partial}{\partial \phi} \right) \psi_{nm} &= \left(-\rho + 2\frac{m+s}{\rho} \right) \psi_{nm} \\ &\quad + \frac{2}{\rho} A_{nm} e^{-\rho^2/2} (\rho e^{i\phi})^{m+s} x \frac{d}{dx} L_n^{m+s}(x). \end{aligned} \quad (167)$$

Then, using the identity [9]

$$\frac{d}{dx} L_a^b(x) = -L_{a-1}^{b+1}(x) \quad (168)$$

for the Laguerre functions, we calculate

$$a_+ \psi_{nm}(\rho, \phi) = \frac{1}{2} e^{i\phi} \left(\rho + \frac{\partial}{\partial \rho} + \frac{i}{\rho} \frac{\partial}{\partial \phi} \right) \psi_{nm} \quad (169)$$

$$= e^{i\phi} \frac{1}{\rho} A_{nm} e^{-\rho^2/2} (\rho e^{i\phi})^{m+s} x \frac{d}{dx} L_n^{m+s}(x) \quad (170)$$

$$= A_{nm} e^{-\rho^2/2} (\rho e^{i\phi})^{m+s} e^{i\phi} \frac{\rho^2}{\rho} (-L_{n-1}^{m+s+1}) , \quad n > 0 \quad (171)$$

$$= -A_{nm} e^{-\rho^2/2} (\rho e^{i\phi})^{m+s+1} L_{n-1}^{m+s+1} , \quad n > 0. \quad (172)$$

Using (89) for A_{nm} we obtain

$$a_+ \psi_{nm} = (-1)^{n+1} \sqrt{\frac{\Gamma(n+1)}{\pi \Gamma(n+m+s+1)}} L_{n-1}^{m+s+1}(x) (\rho e^{i\phi})^{m+s+1} e^{-\frac{\rho^2}{2}} = \sqrt{n} \psi_{n-1, m+1} \quad (173)$$

as required by the first of (120). The second lowering operator acts as

$$a_- \psi_{nm}(\rho, \phi) = \frac{1}{2} e^{-i\phi} \left(\rho + \frac{\partial}{\partial \rho} - \frac{i}{\rho} \frac{\partial}{\partial \phi} \right) \psi_{nm} \quad (174)$$

$$= \frac{m+s}{\rho} e^{-i\phi} \psi_{nm} + e^{-i\phi} \frac{1}{\rho} A_{nm} e^{-\rho^2/2} (\rho e^{i\phi})^{m+s} x \frac{d}{dx} L_n^{m+s}(x) \quad (175)$$

so the identities [9]

$$x \frac{d}{dx} L_a^b(x) = a L_a^b(x) - (a+b) L_{a-1}^b(x) \quad (176)$$

$$L_a^{b-1}(x) = L_a^b(x) - L_{a-1}^b(x) \quad (177)$$

lead to

$$a_- \psi_{nm}(\rho, \phi) = A_{nm} e^{-\rho^2/2} (\rho e^{i\phi})^{m+s-1} \left[(m+s) L_n^{m+s}(x) + n L_n^{m+s}(x) - (n+m+s) L_{n-1}^{m+s}(x) \right] \quad (178)$$

$$= (n+m+s) A_{nm} e^{-\rho^2/2} (\rho e^{i\phi})^{m+s-1} (L_n^{m+s}(x) - L_{n-1}^{m+s}(x)) \quad (179)$$

$$= (n+m+s) A_{nm} e^{-\rho^2/2} (\rho e^{i\phi})^{m+s-1} L_n^{m+s-1}(x) \quad (180)$$

providing

$$\begin{aligned} a_- \psi_{nm} &= (-1)^n \sqrt{\frac{\Gamma(n+1)(m+s+n)^2}{\pi \Gamma(n+m+s+1)}} e^{-\frac{\rho^2}{2}} L_n^{m+s-1}(x) (\rho e^{i\phi})^{m+s-1} \\ &= \sqrt{n+m+s} \psi_{n,m-1} \end{aligned} \quad (181)$$

as required by the first of (121). The raising operator \bar{a}_+ acts as

$$\bar{a}_+ \psi_{nm}(\rho, \phi) = \frac{1}{2} e^{i\phi} \left[\rho - \left(\frac{\partial}{\partial \rho} + \frac{i}{\rho} \frac{\partial}{\partial \phi} \right) \right] \psi_{nm}(\rho, \phi) \quad (182)$$

$$= \rho e^{i\phi} \psi_{nm} - e^{i\phi} A_{nm} \frac{e^{-\rho^2/2} (\rho e^{i\phi})^{m+s}}{\rho} x \frac{d}{dx} L_n^{m+s}(x) \quad (183)$$

so applying identity (168) and the identities [9]

$$L_a^b(x) = L_{a-1}^b(x) + L_a^{b-1}(x) \quad (184)$$

we calculate

$$\bar{a}_+ \psi_{nm}(\rho, \phi) = \rho e^{i\phi} A_{nm} e^{-\rho^2/2} (\rho e^{i\phi})^{m+s} \left(L_n^{m+s}(x) - \frac{d}{dx} L_n^{m+s}(x) \right) \quad (185)$$

$$= A_{nm} e^{-\rho^2/2} (\rho e^{i\phi})^{m+s+1} (L_n^{m+s}(x) + L_{n-1}^{m+s+1}(x)) \quad (186)$$

$$= A_{nm} e^{-\rho^2/2} (\rho e^{i\phi})^{m+s+1} L_n^{m+s+1}(x) \quad (187)$$

providing

$$\bar{a}_+ \psi_{nm} = (-1)^n \sqrt{\frac{\Gamma(n+1)}{\pi \Gamma(n+m+s+1)}} e^{-\rho^2/2} (\rho e^{i\phi})^{m+s+1} L_n^{m+s+1}(\rho^2) \quad (188)$$

$$= \sqrt{n+m+s+1} \psi_{n,m+1} \quad (189)$$

confirming the second of (120). Finally, the action of the second raising operator \bar{a}_- is

$$\bar{a}_- \psi_{nm}(\rho, \phi) = \frac{1}{2} e^{-i\phi} \left[\rho - \left(\frac{\partial}{\partial \rho} - \frac{i}{\rho} \frac{\partial}{\partial \phi} \right) \right] \psi_{nm}(\rho, \phi) \quad (190)$$

$$= e^{-i\phi} \left[\left(\rho - \frac{m+s}{\rho} \right) \psi_{nm} - \frac{A_{nm}}{\rho} e^{-\rho^2/2} (\rho e^{i\phi})^{m+s} x \frac{d}{dx} L_n^{m+s}(x) \right] \quad (191)$$

so using the identity [9]

$$x \frac{d}{dx} L_a^b(x) = (a+1) L_{a+1}^b(x) - (a+b+1-x) L_a^b(x) \quad (192)$$

and (168) leads to

$$\bar{a}_- \psi_{nm}(\rho, \phi) = -A_{nm} e^{-\rho^2/2} (\rho e^{i\phi})^{m+s-1} \left[(m+s-x) L_n^{m+s}(x) + x \frac{d}{dx} L_n^{m+s}(x) \right] \quad (193)$$

$$= -A_{nm} e^{-\rho^2/2} (\rho e^{i\phi})^{m+s-1} [(n+1) L_{n+1}^{m+s-1}(x)] \quad (194)$$

we confirm

$$\bar{a}_- \psi_{nm} = -(-1)^n \sqrt{\frac{(n+1)^2 \Gamma(n+1)}{\pi \Gamma(n+m+s+1)}} e^{-\rho^2/2} L_{n+1}^{m+s-1}(\rho^2) (\rho e^{i\phi})^{m+s-1} \quad (195)$$

$$= \sqrt{n+1} \psi_{n+1, m-1} \quad (196)$$

so that the solutions (91) belong to the ladder representation for any value of s .

Unlike the angular momentum $M = -i\partial_\phi$, which is diagonal in polar coordinates, the remaining $SU(2)$ generators are most conveniently expressed in Cartesian coordinates

$$\Delta = \bar{a}_+ a_+ + \bar{a}_- a_- = N^1 - N^2 = \frac{1}{2} \left(x^2 - y^2 - \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \quad (197)$$

$$Q = -i(\bar{a}_+ a_+ - \bar{a}_- a_-) = xy - \frac{\partial}{\partial x} \frac{\partial}{\partial y} \quad (198)$$

from which it follows that

$$M\psi_0 = s\psi_0 \quad (199)$$

$$\Delta\psi_0 = s \left[\frac{x^2 + y^2 - s + 1}{(x + iy)^2} \right] \psi_0 \quad (200)$$

$$Q\psi_0 = is \left[\frac{x^2 + y^2 - s + 1}{(x + iy)^2} \right] \psi_0 \quad (201)$$

and again we see that the $SU(2)$ symmetry of the Hamiltonian is spontaneously broken for $s \neq 0$.

Since the operator Δ is diagonal in Cartesian coordinates, its action on the states is of special interest. From (125) and (128) and the commutator

$$\left[(\bar{a}_+ a_+ + \bar{a}_- a_-), (\bar{a}_+)^{\alpha} (\bar{a}_-)^{\beta} \right] = \alpha \bar{a}_+^{\alpha-1} (\bar{a}_-)^{\beta+1} + \beta (\bar{a}_+)^{\alpha+1} \bar{a}_-^{\beta-1} \quad (202)$$

we obtain

$$\Delta\zeta_{\alpha\beta} = \alpha\sqrt{\frac{\beta+1}{s+\alpha}}\zeta_{\alpha-1,\beta+1} + \beta\sqrt{\frac{s+\alpha+1}{\beta}}\zeta_{\alpha+1,\beta-1} + (\bar{a}_+)^{\alpha}(\bar{a}_-)^{\beta}\Delta\psi_0 \quad (203)$$

which cannot be diagonalized unless $s = 0$ because (200) shows that the ground state is not an eigenstate of Δ . Using (145) to construct the $s = 0$ multiplets of for given N , it is easily shown, case by case, that diagonalization of Δ recovers the standard Cartesian description of the oscillator. Since Δ cannot be diagonalized on states with $s \neq 0$, there is no unitary combination of the spherical states $\psi_{nm}^{s \neq 0}$ equivalent to the familiar Cartesian states of the harmonic oscillator.

3.2 Number representation for $D = 3$

To obtain a number representation in $D = 3$, we must simultaneously diagonalize the operators \mathbf{M}^2 and M expressed in terms of creation/annihilation operators $(\bar{a}_+, \bar{a}^3, \bar{a}_-)$ for $O(3)$ and $(\bar{a}_+, \bar{a}^0, \bar{a}_-)$ for $O(2,1)$. As seen above, the states defined through the actions of these operators on the ground states diagonalize M . However, the Casimir operators

$$\mathbf{M}^2 = \frac{1}{2}M^{\mu\nu}M_{\mu\nu} = -\frac{1}{2}(\bar{a}^{\mu}a^{\nu} - \bar{a}^{\nu}a^{\mu})(\bar{a}_{\mu}a_{\nu} - \bar{a}_{\nu}a_{\mu}) = N^2 + N - (\bar{a} \cdot \bar{a})(a \cdot a) \quad (204)$$

with total mode number

$$N = \begin{cases} \bar{a}^1 a^1 + \bar{a}^2 a^2 + \bar{a}^3 a^3 = \bar{a}_+ a_- + \bar{a}_- a_+ + \bar{a}^3 a^3, & O(3) \\ \bar{a}^1 a^1 + \bar{a}^2 a^2 + \bar{a}^0 a^0 = \bar{a}_+ a_- + \bar{a}_- a_+ - \bar{a}^0 a^0, & O(2,1) \end{cases} \quad (205)$$

and scalar products

$$\bar{a} \cdot \bar{a} = \begin{cases} 2\bar{a}_+ \bar{a}_- + \bar{a}^3 \bar{a}^3, & O(3) \\ 2\bar{a}_+ \bar{a}_- - \bar{a}^0 \bar{a}^0, & O(2,1) \end{cases} \quad (206)$$

$$a \cdot a = \begin{cases} 2a_+ a_- + a^3 a^3, & O(3) \\ 2a_+ a_- - a^0 a^0, & O(2,1) \end{cases} \quad (207)$$

remain non-diagonal. Nevertheless, \mathbf{M}^2 must be block diagonal with respect to N and M and so studying the expected multiplicity of the mass/energy states leads to a characterization of the oscillator states. Recall that despite the sign of the term $-\bar{a}^0 a^0$ in the second of (205), the total mass/energy of the $O(2,1)$ oscillator was found in equation (63) to be positive definite — because solutions (67) and (68) are of the Feynman, Kislinger, and Ravndal type

for which (16) requires $n^0 \leq 0$, the timelike modes contribute positive mass/energy. We therefore expect that the $O(3)$ and $O(2,1)$ will have similar multiplicity structure, which is verified by examining the wavefunctions as representations of the respective symmetry groups.

The $O(3)$ wavefunctions (65) for $s = 0$ depend on θ and ϕ through the spherical harmonics

$$Y_l^m(\theta, \phi) = C_{lm} P_l^m(\cos \theta) e^{im\phi} \quad (208)$$

and thus provide the familiar $(2l + 1)$ -dimensional representation of $O(3)$ through

$$\mathbf{L}^2 Y_l^m(\theta, \phi) = l(l + 1) Y_l^m(\theta, \phi) \quad \mathbf{L}^3 Y_l^m(\theta, \phi) = m Y_l^m(\theta, \phi) \quad (209)$$

$$L^\pm Y_l^m(\theta, \phi) = \sqrt{(l \mp m)(l \pm m + 1)} Y_{l \pm 1}^m(\theta, \phi) \quad (210)$$

with allowed values

$$l = 0, 1, \dots \quad m = -l, -l + 1, \dots, l - 1, l. \quad (211)$$

Similarly, the $O(2,1)$ wavefunctions (67) for $s = 0$ depend on β and ϕ through the functions

$$\hat{Y}_l^m(\beta, \phi) = C_{lm} \hat{P}_l^m(\sinh \beta) e^{im\phi} \quad (212)$$

and it follows from (28) and (29) that

$$\mathbf{L}^2 \psi_{nlm}^{O(2,1),s=0} = l(l + 1) \psi_{nlm}^{O(2,1),s=0} \quad M \psi_{nlm}^{O(2,1),s=0} = m \psi_{nlm}^{O(2,1),s=0}. \quad (213)$$

The remaining $O(2,1)$ boost generators are

$$A^\pm = A^1 \pm iA^2 = -ie^{\pm i\phi} \left(\partial_\beta \pm i \frac{\sinh \beta}{\cosh \beta} \partial_\phi \right) \quad (214)$$

and for $s = 0$, where we set $\zeta = \sinh \beta$, they take the form

$$A^\pm = -i \frac{e^{\pm i\phi}}{(1 + \zeta^2)^{1/2}} \left[(1 + \zeta^2) \partial_\zeta \pm i\zeta \partial_\phi \right] \quad (215)$$

and

$$A^\pm \hat{Y}_l^m(\beta, \phi) = -i C_{lm} \frac{e^{i(m \pm 1)\phi}}{(1 + \zeta^2)^{1/2}} \left[(1 + \zeta^2) \partial_\zeta \hat{P}_l^m(\zeta) \mp m \zeta \hat{P}_l^m(\zeta) \right]. \quad (216)$$

Using the identities [9]

$$(1 + \zeta^2) \frac{d}{d\zeta} \hat{P}_\nu^\mu(\zeta) = \sqrt{1 + \zeta^2} \hat{P}_{\nu+1}^{\mu+1}(\zeta) + \mu \zeta \hat{P}_\nu^\mu(\zeta) \quad (217)$$

$$(1 + \zeta^2) \frac{d}{d\zeta} \hat{P}_\nu^\mu(\zeta) = (\nu - \mu + 1)(\nu + \mu) \sqrt{1 + \zeta^2} \hat{P}_\nu^{\mu-1}(\zeta) - \mu \zeta \hat{P}_\nu^\mu(\zeta) \quad (218)$$

we find

$$A^+ \hat{Y}_l^m(\beta, \phi) = -i C_{lm} \frac{e^{i(m+1)\phi}}{(1 + \zeta^2)^{1/2}} \left(\sqrt{1 + \zeta^2} \hat{P}_l^{m+1}(\zeta) + m\zeta \hat{P}_l^m(\zeta) - m\zeta \hat{P}_l^m(\zeta) \right) \quad (219)$$

$$= -i C_{lm} e^{i(m+1)\phi} \hat{P}_l^{m+1}(\zeta) \quad (220)$$

$$A^- \hat{Y}_l^m(\beta, \phi) = -i C_{lm} \frac{e^{i(m-1)\phi}}{(1 + \zeta^2)^{1/2}} \left((l - m + 1)(l + m) \sqrt{1 + \zeta^2} \hat{P}_l^{m-1}(\zeta) - m\zeta \hat{P}_l^m(\zeta) + m\zeta \hat{P}_l^m(\zeta) \right) \quad (221)$$

$$= -i(l - m + 1)(l + m) C_{lm} e^{i(m-1)\phi} \hat{P}_l^{m-1}(\zeta) \quad (222)$$

and so that the boost operators A^\pm raise and lower the m eigenvalue as

$$A^\pm \hat{Y}_l^m(\beta, \phi) = \sqrt{(l \mp m)(l \pm m + 1)} \hat{Y}_l^{m \pm 1}(\beta, \phi) \quad (223)$$

comparable to the action of L^\pm in (210). It follows from (213) and (223) that the hyperangular functions (67) provide a $(2l + 1)$ -dimensional representation of $O(2,1)$ with the same multiplicity structure found in the $s = 0$ solutions for $O(3)$. Since the unitary representations of the non-compact group $O(2,1)$ should be infinite-dimensional, the $s = 0$ solutions appear to violate unitarity.

The $s = 1/2$ wavefunctions (66) for $O(3)$ depend on θ and ϕ through the angular functions

$$\hat{\chi}_m^l(\theta, \phi) = F_m^l(z) e^{i(m+1/2)\phi} = C_{lm} (1 + z^2)^{\frac{1}{4}} \hat{P}_m^l(z) e^{i(m+1/2)\phi} \quad (224)$$

where $z = \cot \theta$, and it follows from (27), (49) and (50) that

$$\mathbf{L}^2 \hat{\chi}_m^l(\theta, \phi) = (l^2 - 1/4) \hat{\chi}_m^l(\theta, \phi) \quad M \hat{\chi}_m^l(\theta, \phi) = (m + 1/2) \hat{\chi}_m^l(\theta, \phi). \quad (225)$$

Similarly, the $s = 1/2$ wavefunctions (68) for $O(2,1)$ depend on β and ϕ through the hyperangular functions

$$\chi_m^l(\beta, \phi) = G_m^l(\zeta) e^{i(m+1/2)\phi} = C_{lm} (1 - \zeta^2)^{\frac{1}{4}} P_m^l(\zeta) e^{i(m+1/2)\phi} \quad (226)$$

where $\zeta = \tanh \beta$ and

$$\mathbf{L}^2 \chi_m^l(\beta, \phi) = (l^2 - 1/4) \chi_m^l(\beta, \phi) \quad M \chi_m^l(\beta, \phi) = (m + 1/2) \chi_m^l(\beta, \phi). \quad (227)$$

In terms of the parameters (37) and (38) for $s = 1/2$, the non-diagonal operators for $O(3)$ and $O(2,1)$ take the forms

$$L^\pm = L^1 \pm iL^2 = e^{\pm i\phi} [\pm (1 + z^2) \partial_z - iz\partial_\phi] \quad O(3) \quad (228)$$

$$A^\pm = A^1 \pm iA^2 = e^{\pm i\phi} [-i(1 - \zeta^2) \partial_\zeta \pm \zeta \partial_\phi] \quad O(2,1). \quad (229)$$

For $O(3)$

$$L^\pm \hat{\chi}_m^l(\theta, \phi) = C_{lm} e^{\pm i\phi} [\pm (1 + z^2) \partial_z + (m + 1/2) z] (1 + z^2)^{\frac{1}{4}} \hat{P}_m^l(z) e^{i(m+1/2)\phi} \quad (230)$$

where

$$(1 + z^2) \partial_z (1 + z^2)^{\frac{1}{4}} \hat{P}_m^l(z) = (1 + z^2)^{\frac{1}{4}} \left[\frac{1}{2} z \hat{P}_m^l(z) + (1 + z^2) \frac{d}{dz} \hat{P}_m^l(z) \right] \quad (231)$$

so that

$$L^\pm \hat{\chi}_m^l(\theta, \phi) = C_{lm} e^{i(m+1/2 \pm 1)\phi} (1 + z^2)^{\frac{1}{4}} \left[\pm (1 + z^2) \frac{d}{dz} + (m + 1/2 \pm 1/2) z \right] \hat{P}_m^l(z) \quad (232)$$

and using the identities [9]

$$(1 + z^2) \frac{d}{dz} \hat{P}_\nu^\mu = -(\mu - \nu - 1) \hat{P}_{\nu+1}^\mu - (\nu + 1) z \hat{P}_\nu^\mu = (\mu + \nu) \hat{P}_{\nu-1}^\mu + \nu z \hat{P}_\nu^\mu \quad (233)$$

one is led to

$$\begin{aligned} L^+ \hat{\chi}_m^l(\theta, \phi) &= C_{lm} e^{i(m+1/2+1)\phi} (1 + z^2)^{\frac{1}{4}} \left[-(l - m - 1) \hat{P}_{m+1}^l - (m + 1) z \hat{P}_m^l \right. \\ &\quad \left. + (m + 1) z \hat{P}_m^l(z) \right] \end{aligned} \quad (234)$$

$$= -(l - m - 1) C_{lm} e^{i(m+1/2+1)\phi} (1 + z^2)^{\frac{1}{4}} \hat{P}_{m+1}^l \quad (235)$$

and

$$\begin{aligned} L^- \hat{\chi}_m^l(\theta, \phi) &= C_{lm} e^{i(m+1/2-1)\phi} (1 + z^2)^{\frac{1}{4}} \left[-((l + m) \hat{P}_{m-1}^l + m z \hat{P}_m^l) \right. \\ &\quad \left. + m z \hat{P}_m^l(z) \right] \end{aligned} \quad (236)$$

$$= -(l + m) C_{lm} e^{i(m+1/2-1)\phi} (1 + z^2)^{\frac{1}{4}} \hat{P}_{m-1}^l. \quad (237)$$

The actions of L^\pm and a similar calculation for A^\pm using the identities [9]

$$(1 - \zeta^2) \frac{d}{d\zeta} P_\nu^\mu = (\mu - \nu - 1) P_{\nu+1}^\mu + (\nu + 1) \zeta P_\nu^\mu = (\mu + \nu) P_{\nu-1}^\mu - \nu \zeta P_\nu^\mu \quad (238)$$

leads to

$$L^\pm \hat{\chi}_m^l(\theta, \phi) = \hat{c}(l, m) \hat{\chi}_{m\pm 1}^l(\theta, \phi) \quad A^\pm \chi_m^l(\beta, \phi) = c(l, m) \chi_{m\pm 1}^l(\beta, \phi) \quad (239)$$

where $\hat{c}(l, m)$ and $c(l, m)$ are combinations of the eigenvalues. Since L^\pm and A^\pm act on the lower index (the associated Legendre functions P_ν^μ and \hat{P}_ν^μ are nonzero for $\nu \geq 0$ and $\nu \geq |\mu|$), there is no upper bound on the action of the raising operators L^+ and A^+ , but from

$$\hat{P}_n^n = \frac{(2n)!}{2^n n!} (1 + z^2)^{n/2} \quad P_n^n(\zeta) = (-1)^n \frac{(2n)!}{2^n n!} (1 + \zeta^2)^{n/2} \quad (240)$$

we find the lower bounds

$$L^- \hat{\chi}_m^m(\theta, \phi) = A^- \chi_m^m(\beta, \phi) = 0. \quad (241)$$

The functions F and G therefore provide infinite-dimensional representations of $O(3)$ and $O(2,1)$, leading to mass/energy states of infinite degeneracy, appropriate to the non-compact $O(2,1)$ but apparently violating unitarity for $O(3)$.

Since the multiplicity structure of the wavefunctions (65) to (68) depends on s but not on the relevant symmetry group, we study their eigenvalue content together. We know that for the standard Cartesian states,

$$[M, N^1] \neq 0 \quad [M, N^2] \neq 0 \quad [N, N^\parallel] = [M, N^\parallel] = 0 \quad (242)$$

where the longitudinal component, relative to the choice of $x-y$ plane as locus of observable angular momentum, is

$$N^\parallel = \begin{cases} N^3 & O(3) \\ N^0 & O(2,1) \end{cases}. \quad (243)$$

Therefore, the matrix representation of \mathbf{M}^2 reduces to coherent subspaces labeled by eigenvalues N and n^\parallel , and a convenient parameterization of Cartesian states is

$$\begin{pmatrix} n^1 \\ n^2 \\ n^\parallel \end{pmatrix} = \begin{pmatrix} k \\ N - n^\parallel - k \\ n^\parallel \end{pmatrix} \quad (244)$$

with

$$n^\parallel = 0, 1, \dots, N, \quad k = 0, 1, \dots, (N - n^\parallel). \quad (245)$$

The number of states for given N and n^{\parallel} is therefore $N - n^{\parallel} + 1$, and the total number of states with mode number N is

$$\sum_{n^{\parallel}=0}^N (N + 1 - n^{\parallel}) = (N + 1) (N + 1) - \frac{N (N + 1)}{2} = \frac{(N + 1) (N + 2)}{2} . \quad (246)$$

For $s = 0$, we extend (125) and construct excited states through

$$\zeta_{\alpha\beta\gamma} = \frac{1}{\sqrt{\alpha!\beta!\gamma!}} (\bar{a}_+)^{\alpha} (\bar{a}_-)^{\beta} (\bar{a}^{\parallel})^{\gamma} \psi_0 \quad (247)$$

which are eigenstates of N and M with

$$N\zeta_{\alpha\beta\gamma} = (\alpha + \beta + \gamma) \zeta_{\alpha\beta\gamma} \quad M\zeta_{\alpha\beta\gamma} = (\alpha - \beta) \zeta_{\alpha\beta\gamma} \quad (248)$$

so that the states $\zeta_{\alpha\beta\gamma}$ are precisely the states found by diagonalizing M in the Cartesian picture. Acting on (247) with (204) leads to

$$\begin{aligned} \mathbf{M}^2 \zeta_{\alpha\beta\gamma} &= [N (N + 1) - 4\alpha\beta - \gamma (\gamma - 1)] \zeta_{\alpha\beta\gamma} \\ &\quad - 2\sqrt{(\alpha + 1) (\beta + 1) \gamma (\gamma - 1)} \zeta_{(\alpha+1)(\beta+1)(\gamma-2)} \\ &\quad - 2\sqrt{\alpha\beta (\gamma + 2) (\gamma + 1)} \zeta_{(\alpha-1)(\beta-1)(\gamma+2)} \end{aligned} \quad (249)$$

so that the states $\zeta_{\alpha\beta\gamma}$ are not generally eigenstates of \mathbf{M}^2 , but as expected are mixtures of states with $(\alpha \pm 1, \beta \pm 1, \gamma \mp 2)$ and fixed M eigenvalue

$$m = (\alpha \pm 1) - (\beta \pm 1) = \alpha - \beta. \quad (250)$$

It follows from (249) that

$$\mathbf{M}^2 \zeta_{N00} = N (N + 1) \zeta_{N00} \quad M\zeta_{N00} = N\zeta_{N00} \quad (251)$$

$$\mathbf{M}^2 \zeta_{0N0} = N (N + 1) \zeta_{0N0} \quad M\zeta_{0N0} = -N\zeta_{0N0} \quad (252)$$

and so the allowed eigenvalues of M

$$m = \alpha - \beta = -l, -l + 1, \dots, l - 1, l \quad (253)$$

are consistent with the parameter range

$$\alpha, \beta = 0, 1, \dots, N. \quad (254)$$

Generally, as demonstrated in [6] by exploiting the invariance of $\text{tr}(\mathbf{M}^2)$ under unitary transformations, the Casimir content of the states $\zeta_{\alpha\beta\gamma}$ is

$$l = N, N - 2, \dots, N - \text{int}(N/2) \quad (255)$$

and since the multiplicity of l -states is $2l + 1$, the multiplicity of states with total mode number N is

$$\sum_{k=0}^{\text{int}(N/2)} 2(N - 2k) + 1 = \frac{(N + 1)(N + 2)}{2} \quad (256)$$

in agreement with (246). Since diagonalization of \mathbf{M}^2 does not mix states of different m , states ψ_{lm} have mode number N that depends on l , with Casimir eigenvalues given in (255), but not on m , so there must be a principal quantum number n that complements the contribution of l to energy, incrementing by 2 when l is decremented by 1. Thus, the mode number can be written

$$N = 2n + l, \quad n = 0, 1, 2, \dots, N \quad (257)$$

and the total energy must be

$$E = \omega \left(2n + l + \frac{3}{2} \right) \quad (258)$$

in agreement with the solution (63) to the Schrodinger equation.

According to (247) and (248) the $N = 1$ states constitute the $l = 1$ vector multiplet

$$\zeta^{(1)} = \begin{pmatrix} \zeta_{001} \\ \zeta_{010} \\ -\zeta_{100} \end{pmatrix} = \begin{pmatrix} \bar{a}_- \\ \bar{a}_\parallel \\ -\bar{a}_+ \end{pmatrix} \psi_0^{s=0}, \quad (259)$$

which we order according to the eigenvalues $m = -1, 0, 1$ found by diagonalizing M on the $N = 1$ multiplet of Cartesian states

$$\varphi^{(1)} = \begin{pmatrix} \varphi_{100} \\ \varphi_{010} \\ \varphi_{001} \end{pmatrix} = \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_\parallel \end{pmatrix} \varphi_0. \quad (260)$$

Applying the creation/annihilation operators in polar parameterizations (20)

$$\bar{a}_\pm = \frac{1}{2} e^{\pm i\phi} \left(\rho \sin \theta - \sin \theta \partial_\rho - \frac{\cos \theta}{\rho} \partial_\theta \mp \frac{i}{\rho \sin \theta} \partial_\phi \right) \quad (261)$$

$$\bar{a}^3 = \frac{1}{\sqrt{2}} \left(\rho \cos \theta - \cos \theta \partial_\rho + \frac{\sin \theta}{\rho} \partial_\beta \right) \quad (262)$$

for O(3) and

$$\bar{a}_\pm = \frac{1}{2} e^{\pm i\phi} \left(\rho \cosh \beta - \cosh \beta \partial_\rho + \frac{\sinh \beta}{\rho} \partial_\beta \mp \frac{i}{\rho \cosh \beta} \partial_\phi \right) \quad (263)$$

$$\bar{a}^0 = \frac{1}{\sqrt{2}} \left(\rho \sinh \beta - \sinh \beta \partial_\rho + \frac{\cosh \beta}{\rho} \partial_\beta \right) \quad (264)$$

for O(2,1) to the ground states,

$$\psi_0^{\text{O}(3),s=0} = \psi_0^{\text{O}(2,1),s=0} = A_0 e^{-\rho^2/2} \quad (265)$$

we obtain

$$\zeta^{(1)} = A_0 \rho \begin{pmatrix} \sin \theta e^{-i\phi} \\ \sqrt{2} \cos \theta \\ -\sin \theta e^{i\phi} \end{pmatrix} e^{-\rho^2/2} \quad \text{O}(3) \quad (266)$$

$$\zeta^{(1)} = A_0 \rho \begin{pmatrix} \cosh \beta e^{-i\phi} \\ \sqrt{2} \sinh \beta \\ -\cosh \beta e^{i\phi} \end{pmatrix} e^{-\rho^2/2} \quad \text{O}(2,1). \quad (267)$$

Wavefunctions (266) and (267) are seen to agree with the $l = 1$ vector multiplet found from (65) and (67) using

$$P_1^1(z) = -\sqrt{1-z^2} \quad P_1^0(z) = z \quad P_1^{-1}(z) = \sqrt{1-z^2} \quad (268)$$

$$\hat{P}_1^1(\zeta) = -\sqrt{1+\zeta^2} \quad \hat{P}_1^0(\zeta) = \zeta \quad \hat{P}_1^{-1}(\zeta) = \sqrt{1+\zeta^2}. \quad (269)$$

The $l = 1$ multiplet of the spherical harmonics $Y_l^m(\theta, \phi)$ and $\hat{Y}_l^m(\beta, \phi)$ have the well-known property that the three components form a unit vector, so

$$\rho \begin{pmatrix} Y_1^{-1} \\ Y_1^0 \\ Y_1^1 \end{pmatrix} = \begin{pmatrix} x_- \\ x^3 \\ -x_+ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x - iy \\ \sqrt{2} z \\ -x - iy \end{pmatrix} \quad \text{O}(3) \quad (270)$$

$$\rho \begin{pmatrix} \hat{Y}_1^{-1} \\ \hat{Y}_1^0 \\ \hat{Y}_1^1 \end{pmatrix} = \begin{pmatrix} x_- \\ x^0 \\ -x_+ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x - iy \\ \sqrt{2} t \\ -x - iy \end{pmatrix} \quad \text{O}(2,1) \quad (271)$$

in the basis that diagonalizes the 3×3 matrix representation of M , which may be verified using the parameterizations (20).

The first level of excited states was found by acting on the ground state with the operator multiplet $(\bar{a}_-, \bar{a}^\parallel, -\bar{a}_+)$ which we regard as the fundamental representation of a set of irreducible tensor operators constructed successively by taking irreducible tensor products

$$\bar{a}_m^{(j\pm 1)} = \sum_{m_2=-1,0,1} \langle j \ m - m_2 \ 1 \ m_2 \mid j \ 1 \ j \pm 1 \ m \rangle \bar{a}_{m-m_2}^{(j)} \bar{a}_{m_2}^{(1)} \quad (272)$$

where $\bar{a}_m^{(j\pm 1)}$ is an irreducible tensor operator of rank $j \pm 1$, $\bar{a}_{m_2}^{(1)}$ is the vector operator, and $\langle j \ m - m_2 \ 1 \ m_2 \mid j \ 1 \ j \pm 1 \ m \rangle$ is the appropriate Clebsch-Gordan coefficient. Thus, according to (246), the $N = 2$ states have total multiplicity of 6, which by (255) must include the five $l = 2$ states and the $l = 0$ singlet state. The two irreducible tensor operators that can be constructed from the vector operator are the singlet ($l = m = 0$)

$$\bar{a}_0^{(0)} = -\frac{1}{\sqrt{3}} (2\bar{a}_+ \bar{a}_- + \bar{a}_3 \bar{a}_3) = -\frac{1}{\sqrt{3}} \bar{a} \cdot \bar{a} \quad (273)$$

and the $l = 2$ operators

$$\bar{a}_{-2}^{(2)} = \bar{a}_{-1}^{(1)} \bar{a}_{-1}^{(1)} = (\bar{a}_-)^2 \quad (274)$$

$$\bar{a}_{-1}^{(2)} = \frac{1}{\sqrt{2}} \left(\bar{a}_0^{(1)} \bar{a}_{-1}^{(1)} + \bar{a}_{-1}^{(1)} \bar{a}_0^{(1)} \right) = -\sqrt{2} \bar{a}_- \bar{a}_3 \quad (275)$$

$$\bar{a}_0^{(2)} = \frac{1}{\sqrt{6}} \left(\bar{a}_1^{(1)} \bar{a}_{-1}^{(1)} + 2\bar{a}_0^{(1)} \bar{a}_0^{(1)} + \bar{a}_{-1}^{(1)} \bar{a}_1^{(1)} \right) = -\frac{2}{\sqrt{6}} (\bar{a}_+ \bar{a}_- - \bar{a}_3 \bar{a}_3) \quad (276)$$

$$\bar{a}_1^{(2)} = \frac{1}{\sqrt{2}} \left(\bar{a}_1^{(1)} \bar{a}_0^{(1)} + \bar{a}_{-1}^{(1)} \bar{a}_1^{(1)} \right) = \sqrt{2} \bar{a}_+ \bar{a}_3 \quad (277)$$

$$\bar{a}_2^{(2)} = \bar{a}_1^{(1)} \bar{a}_1^{(1)} = (\bar{a}_+)^2 \quad (278)$$

which are precisely the operators found by diagonalizing the matrix representation of \mathbf{L}^2 . In this way, the complete set of spherical polar harmonic oscillators in 3 dimensions can be constructed from the ground state.

For the $s = 1/2$ wave functions, the attempt to build excited states from the ground state and the operator multiplet $(\bar{a}_-, \bar{a}^\parallel, -\bar{a}_+)$ fails immediately. The infinitely degenerate ground states found from (66) and (68) are

$$\psi_0 = A e^{-\rho^2/2} \rho^{-1/2} (1 + z^2)^{\frac{1}{4}} \hat{P}_m^0(z) e^{i(m+\frac{1}{2})}, \quad m = 0, 1, 2, \dots, \quad \text{O}(3) \quad (279)$$

$$\psi_0 = A e^{-\rho^2/2} \rho^{-1/2} (1 - \zeta^2)^{\frac{1}{4}} P_m^0(\zeta) e^{i(m+\frac{1}{2})}, \quad m = 0, 1, 2, \dots, \quad \text{O}(2,1) \quad (280)$$

and the action of the vector multiplet of creation operators on these states generates complicated functions that do not even approximate the first excited levels. Apparently, the vector operator multiplet belongs only to the $s = 0$ vector representations of the symmetry groups, and not to the infinite-dimensional $s = 1/2$ representations. It may be possible to construct an appropriate ladder representation of creation/annihilation operators though a multipole expansion of the Hamiltonian, corresponding to an infinite summation of the associated Legendre functions. This will be discussed in a subsequent paper.

A working mode number representation for the $s = 1/2$ representations of $O(2,1)$ is required to clarify the question of ghost states for the relativistic harmonic oscillator. As seen in (18) excited timelike modes of the Feynman, Kislinger, and Ravndal wavefunctions may have negative norm, which were handled in [8] by applying the covariant condition

$$(p \cdot a) \psi = 0 \quad (281)$$

which forces ψ into the ground state along the momentum p , suppressing timelike excitations. Although this approach is comparable to Gupta-Bleuler quantization [10] of the electromagnetic field, where (281) expresses the Lorentz gauge condition as an operator equation and eliminates negative norm states along the field's lightlike momentum, there is no applicable gauge condition for the general relativistic oscillator that justifies this procedure.

Interestingly, the problem of negative normed states could have been inadvertently overlooked without reference to the creation/annihilation operators, because without sufficient attention to the properties of the states as representations of the Lorentz group, we might neglect to include the metric in our calculations. For example, the first excited timelike mode is

$$\psi_1^0 = \bar{a}^0 \psi_0 = \frac{1}{\sqrt{2}} (x^0 - \partial^0) A_0 e^{-\rho^2/2} = A_0 \sqrt{2} t e^{-\rho^2/2} \quad (282)$$

and we may follow the method of (18) to calculate

$$\begin{aligned} \int dt d^2x |\psi_1^0|^2 &= \int dt d^2x |\bar{a}^0 \psi_0|^2 = \frac{1}{2} \int dt d^2x ((x^0 - \partial^0) \psi_0)^\dagger ((x^0 - \partial^0) \psi_0) \\ &= \int dt d^2x \frac{1}{2} \psi_0^* (x^0 + \partial^0) (x^0 - \partial^0) \psi_0 \\ &= \int dt d^2x \frac{1}{2} \psi_0^* [(x^0 - \partial^0) (x^0 + \partial^0) + 2\eta^{00}] \psi_0 \\ &= - \int dt d^2x \psi_0^* \psi_0 < 0 \end{aligned} \quad (283)$$

where we use

$$(x^0 - \partial_0) \psi_0 = 0 \quad (284)$$

and assume some regularization for the ground state normalization. However, neglecting to include the metric in the formulation of the norm, we might be tempted to calculate

$$\int dt d^2x |\psi_1^0|^2 = \int dt d^2x |A_0 \sqrt{2} t e^{-\rho^2/2}|^2 = 2 \int dt d^2x t^2 \psi_0^* \psi_0 > 0 \quad (285)$$

which contradicts (283). Given the role played by the metric in (283), it seems that the proper formulation of the norm requires that we respect the tensor properties of each excited state and not inadvertently treat the states as scalar entities. Thus, the norm (285) should be restated as

$$\int dt d^2x \|\psi_{nlm}\|^2 = \int dt d^2x \eta_{mm} (\psi_{nlm})^\dagger \psi_{nlm} \quad (286)$$

where η_{mm} represents the metric in the relevant tensor representation. In the absence of a number representation for the relativistic oscillator, the straightforward calculation in (18) cannot be performed to check that no ghosts appear in this formulation. We may argue that since the wavefunctions (68) are not separable into Cartesian modes, all polar modes mix space and time within the spacelike sector, and so there should be no timelike excitations as such in the relativistic oscillator. Moreover, given the infinite dimensional multiplets of states, there is no particular state that is naturally assigned a negative metric. These claims will receive more detailed treatment in a subsequent paper.

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